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Coherence and the electron field

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Abstract. A unified formulation of coherence which is appropriate to both the electromagnetic and electron fields is proposed. The degree of coherence possible for the electron field and its experimental consequences are investigated.

1. Introduction

The concept of coherence occurs widely in physics but its definition appears to vary depending on the experiment considered. The well known classical definitions relevant to many photon optics experiments have been extended by Glauber (1964) to the quantum case (Mandel and Wolf 1965). These definitions are not entirely relevant however to experiments involving interference of independent beams (Mandel 1964a, b) and a more appropriate definition is developed. After an analysis of the common basis underlying all definitions of coherence (including that of Rocca 1973), definitions of electron coherence analogous to those of Glauber are proposed and the implications of these are explored systematically.

2. The coherence concept

Coherence in general appears to be attributed to states which behave like those of some idealized or classical model of the system being studied, in the context of a well defined subset of the totality of experiments capable of being performed on that system. In virtually all cases the idealized model is taken to be a simple wave model and the type or degree of coherence is defined in a way depending on the set of experiments chosen. For example Glauber's widely used criterion for first-order photon coherence

$$\langle A_\mu^-(x_1)A_\nu^+(x_2)\rangle = V_\mu^*(x_1)V_\nu(x_2) \tag{1}$$

for some function $V_\mu(x)$, is equivalent to the following definition which is expressed in terms of an idealized model:

Definition (I). A photon state is said to be coherent if and only if the results in all possible experiments in which only the instantaneous intensity is measured, are such that they are capable of being predicted by appealing to a single wave $\xi_\mu(x)$ (with the intensity given in the usual way by $S_{\mu\nu}\xi_\mu^*(x)\xi_\nu(x)$) without recourse to an averaging process of any kind. The waves are taken to satisfy the electromagnetic wave equation in the region concerned;

$$\Box^2\xi_\mu(x) = 0. \tag{2}$$

The equivalence of these definitions can be shown using the quantum expression for the instantaneous intensity

$$I(x) = S_{\mu\nu}\langle A_\mu^-(x_1)A_\nu^+(x_2)\rangle = S_{\mu\nu} \sum_{k_1k_2} \Phi^*_{k_1,\mu}(x_1)\Phi_{k_2,\nu}(x_2)\langle a^\dagger_{k_1,\mu}a_{k_2,\nu}\rangle. \tag{3}$$

In this expression $a_{k,\mu}$ is the annihilation operator of the state k, μ represented by the time-dependent wavefunction $\Phi_{k,\mu}(x)$. In free space $\Phi_{k,\mu}(x) = C_k\,\mathrm{e}^{\mathrm{i}k.x}$ or in an interference experiment $\Phi_{k,\mu}(x)$ can be taken as the solution to the single-photon equation compatible with the boundary conditions imposed by the slits etc. It can be seen from (3) that a state fulfils definition (I) if and only if the matrix $\langle a^\dagger_{k_1,\mu}a_{k_2,\nu}\rangle$ has only one non-zero eigenvalue λ_μ (with eigenvector $q_{k,\mu}$ say) since, if one defines

$$\xi_\mu(x) = \sum_k q_{k,\mu}\Phi_{k,\mu}(x)\lambda_\mu^{1/2} \tag{4}$$

then (3) is in the form required by definition (I) and $\xi_\mu(x)$ is a solution of the wave equation. The existence of precisely one non-zero eigenvalue is equivalent to the possibility of factorizing $\langle a^\mu_{k_2,\mu}a_{k_2,\nu}\rangle$ into the form $\lambda q^*_{k_1,\mu}q_{k_2,\nu}$ and this in turn is equivalent to requiring that $\langle A_\mu^-(x_1)A_\nu^+(x_2)\rangle$ factorize as in equation (1). Thus the equivalence has been shown.

Under these definitions any incoherence is due entirely to the quantum nature of the field and not to any spread in frequencies or momenta. Any such spread will merely cause rapid fluctuations in the instantaneous intensity without affecting the coherence. This contrasts with the situation in classical optics where a different definition of coherence is used. This classical definition may also be applied to quantum optics and is relevant to experiments which measure the intensity averaged over some time interval.

> *Definition (II).* A state is said to be coherent if and only if the results in all possible experiments, in which only the intensity averaged over a certain time interval is measured, are such that they are capable of being predicted by considering a single frequency wave $\xi_\mu(x)$ obeying equation (2) and without recourse to an averaging process of any kind.

Definition (II) is very convenient as, in practice, intensities are often measured over times long compared to the fluctuations and also since single frequency waves are very easily treated theoretically. With this definition, incoherence arises from any spread in frequencies and momenta as well as from the quantum nature of the state.

Coherent states of any sort are easy to treat theoretically because of the idealized properties they possess, but even if a state is not coherent the concept of a coherent state is very useful. For example, if definition (I) is used we can make use of the Hermitian nature of the matrix $\langle a^\dagger_{k_1,\mu}a_{k_2,\nu}\rangle$. This matrix can be diagonalized giving positive eigenvalues λ^i_μ with associated eigenvectors $q^i_{k,\mu}$. Thus (3) can always be written in the form

$$I(x) = \sum_i S_{\mu\nu}\xi^{i*}_\mu(x)\xi^i_\nu(x) \tag{5}$$

where $\xi^i_\mu(x)$ is defined analogously to (4). Hence the results in intensity measuring experiments can always be predicted on the basis of an ensemble of states which behave in the idealized manner; ie by merely summing the predictions obtained for each of a set of easily treated coherent states. The ensemble of functions $\xi^i_\mu(x)$ is sometimes considered as directly corresponding to the ensemble of particles which are being detected but as can be seen above this is in general only a convenient fiction.

Although the use of ensembles of coherent states is a great simplification, in certain cases the description of states can be made even simpler by defining coherence functions which describe in a more average sense how well the idealized behaviour fits the actual behaviour. The first-order coherence function corresponding to Glauber's definition of coherence (I) is well known but it is perhaps appropriate to comment on the coherence function associated with definition (II). This can be written

$$g^{(\mathrm{II})}_{\mu\nu}(x_1, x_2) = \frac{\overline{\langle A_\mu^-(x_1)A_\nu^+(x_2)\rangle}}{(\overline{\langle A_\mu^-(x_1)A_\mu^+(x_1)\rangle}\,\overline{\langle (A_\nu^-(x_2)A_\nu^+(x_2)\rangle})^{1/2}}$$

where the bar indicates time averaging. Although such time averaging occurs almost universally in experiments it is seldom if ever included in the definitions of coherence functions. If the interval of time averaging (τ) is much greater than the period of any fluctuations ($1/\Delta\omega$) then the time averaging operation is equivalent to deleting certain terms in the density matrix for the state. Consider the density operator for an arbitrary state

$$\rho = \sum_{\{n_k\}} \sum_{\{m_k\}} P(\{n_k\}, \{m_k\}) |\{n_k\}\rangle\langle\{m_k\}|.$$

Hence

$$\overline{\langle A^-(x_1)A^+(x_2)\rangle} = \sum_{\{n_k\}} \sum_{\{m_k\}} P(\{n_k\}, \{m_k\}) \sum_{k_1} \sum_{k_2} C^*_{k_1} C_{k_2} \langle\{m_k\}|a^\dagger_{k_1} a_{k_2}|\{n_k\}\rangle$$
$$\times \int_0^\infty \exp[+\mathrm{i}(\omega_{k_1} - \omega_{k_2})t]\,\mathrm{d}t.$$

Equating to zero those $P(\{n_k\}, \{m_k\})$ which involve non-degenerate states $|\{n_k\}\rangle$, $\langle\{m_k\}|$, and those which differ by more than one occupation number, yields a new density operator ρ' which will give the time-averaged expectation value without explicitly time averaging. The process is often implicitly included by treating only states which are already of this form (ie stationary states).

Coherence is also frequently defined for experiments which measure intensity correlations rather than simple intensities. Such a definition can easily be arrived at by replacing 'instantaneous intensity' with '*n*th order intensity correlation' in definition (*I*). For such a definition the representation of an arbitrary state by an ensemble of coherent states cannot in general be obtained by diagonalization of a Hermitian matrix and indeed does not always exist in the orthodox sense. However with a generalization of the concept of an ensemble, such a representation is possible by virtue of the optical equivalence theorem (Klauder and Sudarshan 1968). Coherence functions describing the extent to which a state is coherent in the above sense are described elsewhere (Glauber 1964). Another type of coherence which could be of much use in discussing experiments involving interference between independent beams (Mandel 1964a, b) can be defined by replacing in definition (I) 'experiments in which only the instantaneous intensity is measured' with 'experiments involving interference of the beam being studied with another independent beam'. This definition is referred to as *definition* (*III*). The combined beams will behave in the idealized (wavelike) manner if this total beam is coherent according to definition (I). As Glauber has shown, this is equivalent to requiring that for all k_1, k_2, μ, ν

$$|\langle a^\dagger_{k_1,\mu} a_{k_2,\nu}\rangle|^2 = \langle a^\dagger_{k_1,\mu} a_{k_1,\mu}\rangle\langle a^\dagger_{k_2,\nu} a_{k_2,\nu}\rangle. \qquad (6)$$

If k_1 is in beam (1) and k_2 in beam (2) then because the beams are independent equation (6) may be factorized

$$|\langle a^\dagger_{k_1,\mu}\rangle|^2 = \langle a^\dagger_{k_1,\mu} a_{k_1,\mu}\rangle. \tag{7}$$

A state which is coherent in this sense, not only has a well defined relative phase between different points in space as required by definition (I) but has a well defined absolute phase (defined by the phase of $\langle a_{k,\mu}\rangle$). If there are many modes in the beam then it is perhaps more convenient to use an equivalent form in coordinate space, namely,

$$|\langle A^+_\mu(x)\rangle|^2 = \langle A^-_\mu(x) A^+_\mu(x)\rangle. \tag{8}$$

Coherence functions can be defined in the following ways:

$$g^{(\mathrm{III})}_\mu(k) = \frac{\langle a_{k,\mu}\rangle}{\langle a^\dagger_{k,\mu} a_{k,\mu}\rangle^{1/2}}$$

$$g^{(\mathrm{III})}_\mu(x) = \frac{\langle A^+_\mu(x)\rangle}{\langle A^-_\mu(x) A^+_\mu(x)\rangle^{1/2}}.$$

The only states which are fully coherent in this sense are the eigenstates of the annihilation operators which are, as is well known, coherent to all orders. Any stationary state will be completely incoherent using this definition, ie $g^{(\mathrm{III})} \equiv 0$. Thus, although stationary states are useful in the situations outlined in definition (II), in general it is inappropriate to use stationary states in the description of experiments involving interference of independent beams.

3. Electron coherence

Rocca (1973) provides a definition of electron coherence which can be expressed in the manner of § 2.

Definition (IV). An electron state is coherent if and only if the photon counting statistics obtained using a detector in this state are of the same simple form obtained using a semi-classical model or an approximate quantum model, namely a compound Poisson distribution.

Although such a definition could be useful for experiments involving photodetection, in the field of electron optics the following definition related to that used in photon optics is perhaps more applicable and in line with the usual idea of coherence.

Definition (V). An electron state is coherent if and only if the results in all possible experiments in which only the instantaneous 'electron field intensity' is measured, are such that they are capable of being predicted by appealing to a single (spinor) wave $\zeta_i(x)$ (with the intensity given by $S'_{ij}\bar{\zeta}_i(x)\zeta_j(x)$ where the bar indicates the spinor adjoint; $\zeta^*\gamma_4$) without recourse to an averaging process of any kind. The waves are taken to satisfy the Dirac equation;

$$\gamma^{ij}_\mu \frac{\partial \zeta_j}{\partial x_\mu} + M\zeta_i - \mathrm{i}eA_\mu(x)\gamma^{ij}_\mu \zeta_j = 0.$$

In order to apply this definition it is necessary to choose a particular expression for the electron field 'intensity' which is both convenient and physically accurate. From a study of the electron detection process (Every 1973) it would appear that the expression

$$I(x) = S'_{ij}\langle \bar{\psi}_i^-(x)\psi_j^+(x)\rangle$$

would be suitable over at least a small range of electron energies ($\Delta E \ll E$).

Time average electron coherence can also be defined by analogy with definition (II) for photons and this will be designated *definition (VI)*. The remaining definitions of photon coherence are readily carried over into the area of electron optics. Electron coherence of the nth order thus causes

$$\left\langle \prod_{i=1}^{n} \bar{\psi}_{j(i)}^-(x_i) \prod_{i=n}^{1} \psi_{m(i)}^+(x'_i) \right\rangle$$

to factorize. The definition corresponding to definition (III) or equations (7) and (8), for photons and applicable to experiments involving interference of independent beams, will be referred to as *definition (VII)*.

The anticommutation rules which apply to the electron field do however place very severe restrictions on the possibilities for the existence of coherent states (Bowring *et al* 1971). The one-particle states are trivially first-order coherent (under definition (V) and also definition (VI) if they are energy eigenstates) but apart from these no state exists which is coherent in any sense or for any order. This can be shown by an argument the first part of which is based on one by Klauder and Sudarshan (1968). Assume $|\psi\rangle$ is a first-order coherent state. Then, as for photons, this implies that $\langle c_k^\dagger c_l\rangle = q_k^* q_l$ for some sequence $\{q_k\}$. Define $b_1 = \Sigma_l U_{1l} c_l$ where

$$U_{1l} = \frac{q_l^*}{(\Sigma |q_k|^2)^{1/2}}.$$

It can be seen that b_1 is a conventional destruction operator for a new mode or single particle state $\Sigma_l U_{1l}|l\rangle$. The transformation to this new mode can be considered as part of a unitary transformation to a new basis of modes which includes this new mode.

$$b_m = \Sigma_l U_{ml} c_l.$$

It can also be seen that $\langle b_m^+ b_m\rangle = 0$ unless $m = 1$ and from this we can conclude that the state must be entirely of the mode defined by b_1, ie it must be a linear combination of states of the form

$$\frac{(b_1^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

Although for photons this is quite a general state, for electrons $(b_1^\dagger)^n = 0$ unless $n = 1$ and thus only one-electron states will be (trivially) first-order coherent. Higher-order coherent states are impossible since $\langle c_k^\dagger c_l^\dagger c_m c_n\rangle = q_k^* q_l^* q_m q_n$ is necessary and this leads to the contradiction,

$$\langle c_k^\dagger c_l^\dagger c_m c_n\rangle = -\langle c_k^\dagger c_l^\dagger c_n c_m\rangle = -q_k^* q_l^* q_n q_m.$$

Full coherence in the sense of definition (VII) is also impossible since two such beams combined must be first-order coherent and must hence be expressible as a one-particle state which cannot be factorized into independent beams.

However, in spite of the non-existence of non-trivial coherent states we can, for the purpose of describing experiments which measure only intensity, still represent the electron state as an ensemble of first-order coherent states in the manner of equation (5) and we can still define the extent to which a given state approximates any sort or order of coherence by means of coherence functions defined analogously to $g^{(I)}_{\mu\nu}$, $g^{(II)}_{\mu\nu}$ and $g^{(III)}_{\mu}$. For example we can define

$$g^{(V)}_{ij}(x_1, x_2) \equiv \frac{\langle \bar{\psi}_i^-(x_1)\psi_j^+(x_2)\rangle}{(\langle \bar{\psi}_i^-(x_1)\psi_i^+(x_1)\rangle\langle \bar{\psi}_j^-(x_2)\psi_j^+(x_2)\rangle)^{1/2}}.$$

The question arises as to how closely a state can approach coherence with a given measure of coherence and set of constraints. If the instantaneous intensity is measured, then the spread of momenta is of no consequence. Consider a finite set of n momentum states and consider the set of all states which can be formed from these momentum values such that no momentum is favoured over all the rest, ie symmetrical in all momenta. The general state of this form is

$$\begin{aligned} a_0|0\rangle &+ a_1(|0,0,\ldots,0,1\rangle + |0,0,\ldots,1,0\rangle + \ldots + |1,0,\ldots,0,0\rangle) \\ &+ a_2(|0,0,\ldots,1,1\rangle + |0,\ldots,1,0,1\rangle + \ldots + |1,1,\ldots,0,0\rangle) \\ &+ \ldots + a_n|1,1,\ldots,1,1\rangle. \end{aligned} \tag{9}$$

The coherence (in momentum space) is given by

$$g^{(V)}(k_1, k_2) = \frac{\langle c^\dagger_{k_1} c_{k_2}\rangle}{(\langle c^\dagger_{k_1} c_{k_1}\rangle\langle c^\dagger_{k_2} c_{k_2}\rangle)^{1/2}} \tag{10}$$

and it is this function (independent of $k_1 \neq k_2$) which will be maximized by varying the coefficients a_i subject to two constraints. The first constraint is that the state has a fixed average number of electrons, $\bar{N}$ and the second constraint is the normalization of the state. Substituting the state (9) into (10) and using an explicit expression for $\bar{N}$ one obtains

$$g^{(V)}(k_1 \neq k_2) = \frac{n}{\bar{N}} \sum_{m=1}^{n} \frac{(n-2)!}{(m-1)!(n-m-1)!} |a_m|^2 \tag{11}$$

with the normalization constraints

$$\sum_{m=1}^{n} \frac{n!}{(n-m)!m!} |a_m|^2 = 1$$

$$\sum_{m=1}^{n} \frac{n!}{(m-1)!(n-m)!} |a_m|^2 = \bar{N}.$$

Use of Lagrange multipliers and differentiation leads to a homogeneous set of equations which are soluble only when exactly two of the a_i are non-zero coefficients.

Solving for these gives

$$|a_p|^2 = \left(\frac{p!(n-p)!}{n!}\right)\left(\frac{q-\bar{N}}{q-p}\right)$$

$$|a_q|^2 = \left(\frac{q!(n-q)!}{n!}\right)\left(\frac{p-\bar{N}}{p-q}\right).$$

For all values of p and q the corresponding state maximizes $g^{(V)}$ in the local sense. Substituting into (11) gives

$$g^{(V)}(k_1 \neq k_2) = \frac{n-p-q}{n-1} + \frac{pq}{\bar{N}(n-1)}.$$

To illustrate this result figure 1 is a graph of $g^{(V)}(k_1 \neq k_2)$ plotted against $\bar{N}$ for $n = 4$ and all possible values of (p, q). For a given value of $\bar{N}$ the largest value of $g^{(V)}(k_1 \neq k_2)$ can be shown rigorously to occur when the difference between p and q is minimized.

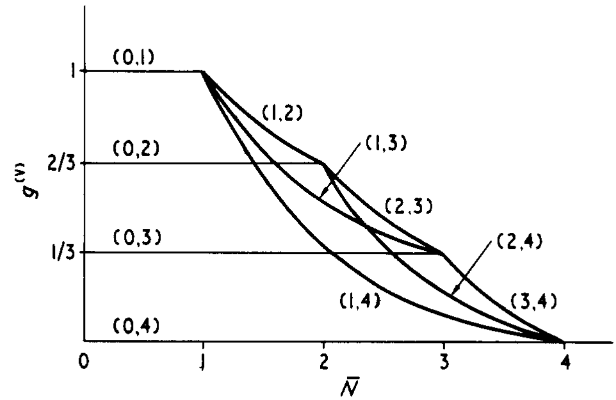

Figure 1. A plot of $g^{(V)}(k_1 \neq k_2)$ against $\bar{N}$ for $n = 4$ and all possible values of (p, q).

For non-integral values of $\bar{N}$, p and q will differ by one and can be taken to satisfy $p < \bar{N} < q$. This gives

$$g^{(V)}(k_1 \neq k_2) = \frac{n-2p-1}{n-1} + \frac{p(p+1)}{\bar{N}(n-1)}.$$

For integral values of $\bar{N}$, $p = q = \bar{N}$ and hence

$$g^{(V)}(\bar{N}) = 1 - \left(\frac{\bar{N}-1}{n-1}\right). \tag{12}$$

From equation (12) it can be seen that for a given $\bar{N}$ the coherence is maximized by taking only one non-vanishing coefficient $a_{\bar{N}}$ and by making n as large as possible, ie by spreading the $\bar{N}$ electrons amongst as many modes as possible. Such states will be referred to as 'dilute' states and loosely speaking have as much of the ground state per mode as is possible.

This sort of 'dilute' state is also the sort of state which is most coherent in the sense of definition (VII). Consider an arbitrary fermion state of a single mode k

$$(1-|\alpha|^2)^{1/2}|0\rangle + \alpha|k\rangle.$$

Then

$$g^{(VII)}(k) = \frac{\langle c_k \rangle}{\langle c_k^\dagger c_k \rangle^{1/2}} = (1-|\alpha|^2)^{1/2}$$

which tends to a maximum value of 1 as $|\alpha| \to 0$. If coherence is maximized keeping

$\bar{N}$ constant the number of modes must be increased as much as possible, ie the most coherent state in this sense is the most 'dilute' state.

More usually it is the intensity averaged over a short time interval which is measured. Thus as discussed in the previous section only stationary states need be considered. If the modes are chosen suitably or if there is sufficient spatial averaging this means that one need only consider ensembles of states of the form $|\boldsymbol{k}_1\ \boldsymbol{k}_2, \boldsymbol{k}_3, \ldots, \boldsymbol{k}_n\rangle$. As mentioned in the previous section such coherence depends on the spread of momenta which cannot be reduced indefinitely for the electron field if there is a finite density of states due to the fact that each pair of electrons present must be in a different mode. Assume an n electron state of the above form which, in order to maximize the coherence, has been concentrated around a particular momentum value ($\boldsymbol{k}_0$ say) as closely as possible. For definiteness assume a distribution, in momentum space of the form of a rectangular prism with dimensions d_x, d_y, d_z, along the appropriate axes which have been chosen such that $\boldsymbol{k}_0$ is parallel to the z axis. For free electrons in a cavity of dimensions L_x, L_y, L_z the linear densities of states along the axes are $L_x/2\pi$, $L_y/2\pi$, $L_z/2\pi$ and thus the maximization criterion gives

$$d_x d_y d_z L_x L_y L_z \geqslant (2\pi)^3 n.$$

The coherence lengths l_x, l_y, l_z are roughly the product of 2π and the reciprocal of the appropriate dimensions d_x, d_y, d_z and thus we get

$$l_x l_y l_z \leqslant \left(\frac{V}{n}\right) = \frac{1}{\rho_v}. \tag{13}$$

Thus the coherence volume cannot be made to exceed the average effective volume per electron in the cavity.

Thus if a beam of electrons is generated in a cavity containing electrons with an effective density of 10^{15} electrons m^{-3} then the average dimension of the maximum coherence volume will be roughly 10^{-5} m. The maximum coherence time τ can be obtained by dividing l_z by the mean electron speed v_0 or relativistically by $k_0(m_e^2 + k_0^2)^{-1/2}$.

Another way of expressing this restriction on maximum coherence is to note that the maximum degeneracy parameter δ is given by

$$\delta \equiv \rho_v l_x l_y l_z = 1.$$

This contrasts with the photon case where, in a maser beam for example, δ may be much greater than unity. Of course at densities as high as 10^{15} electrons m^{-3} space charge and other interaction effects will dominate and increase the spread of momenta but equation (13) still represents a valid upper limit to the coherence volume.

4. Conclusions

Interference experiments with electron beams have been performed by several authors such as Möllenstedt and Düker (1956) and Jönsson (1961) (Jönsson *et al* (1974) is an English version of the latter reference). The results obtained appear to agree at least qualitatively with the wave theory of interference. The considerations of the previous sections have justified use of such a theory combined with a suitable definition of coherence in explaining interference experiments with electron beams. The definitions of coherence proposed here are directly analogous to the usual definitions in photon optics in contrast to the definition proposed by Rocca (1973) which is directly relevant only to the electron states in photon detectors. A state which is coherent under Rocca's definition would be incoherent under the definitions of the previous section. Quantitative estimates of the coherence present in the electron beams used in the experiments of Möllenstedt and Düker do not agree well with simple predictions based on the thermal spread in electron energies (Klemperer 1972). However it is likely that the differences are due to defects in the treatment of the optics of the collimation and magnification process or to defects in the assumptions made concerning the production of and interactions in, the electron beam itself rather than to any fundamental property of electron coherence such as that given by equation (13).

Fundamental differences do exist between electron and photon coherence defined in either the quantum or time-averaged sense due to the difference in the wave equation and (anti)commutation rules for the two fields. These differences have been outlined in the previous section for several definitions of coherence and could be important for possible future experiments on electron beams of high monochromaticity with apparatus capable of resolving the fine details observable in an almost coherent beam. Complete second-order coherence, as mentioned in the last section, cannot be observed for electrons and because of the anticommutation rules cannot even be approached arbitrarily closely. Nevertheless second-order correlation experiments can still, in principle, yield a great deal of information about an electron beam, but at the moment, to the author's knowledge, the temporal resolution and beam monochromaticity available are not sufficient to obtain useful results. As noted elsewhere (Bénard 1970) the theory predicts an anticoincidence effect in contradistinction to the coincidence effect observed with photons and its observation would indeed be a very direct confirmation of the anticommutation relations of the electron field.

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